Axioms of Plane Geometry 2

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Undefined Terms: Point, Line, Distance, Half-plane, Angle measure, Area

**Axiom 5.3.1 (The Existence Postulate):** The collection of all points forms a *nonempty* set. There is *more than one* point in that set.

**Axiom 5.3.3 (The Incidence Postulate):** Every line is a set of points. For every pair of distinct points $A$ and $B$, there is exactly one line $\ell$ such that $A \in \ell$ and $B \in \ell$.

**Axiom 5.4.1 (The Ruler Postulate):** For every pair of points $P$ and $Q$, there exists a real number $PQ$, called the distance from $P$ to $Q$. For each line $\ell$, there is a one-to-one correspondence from $\ell$ to $\mathbb{R}$ such that if $P$ and $Q$ are points on the line that correspond to the real numbers $x$ and $y$, respectively, then $PQ = |x - y|$. 
Def: Plane, lie on, external point, parallel, segment, ray, length, congruent.

Def: Let $A$, $B$, and $C$ be three distinct points. The point $C$ is *between* $A$ and $B$, written $A \ast C \ast B$, if $C \in \overrightarrow{AB}$ and $AC + CB = AB$.

Theorem 5.3.7: If $\ell$ and $m$ are two distinct, nonparallel lines, then there exists exactly one point $P$ such that $P$ lies on both $\ell$ and $m$. 
Theorem (Theorem 5.4.6:)

If $P$ and $Q$ are any two points, then

1. $PQ = QP$,
2. $PQ \geq 0$, and
3. $PQ = 0$ if and only if $P = Q$.

Corollary (Corollary 5.4.7:)

$A \ast C \ast B$ if and only if $B \ast C \ast A$. 
Def: A (semi-)metric is a function $D : \mathbb{P} \times \mathbb{P} \rightarrow \mathbb{R}$ such that

1. $D(P, Q) = D(Q, P)$ for every $P$ and $Q$,
2. $D(P, Q) \geq 0$ for every $P$ and $Q$, and
3. $D(P, Q) = 0$ if and only if $P = Q$.

Ex (Euclidean): $d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
Ex (Taxicab): $\rho((x_1, y_1), (x_2, y_2)) = |x_2 - x_1| + |y_2 - y_1|$.
**Def:** Let $\ell$ be a line. A one-to-one correspondence $f : \ell \to \mathbb{R}$ such that $PQ = |f(P) - f(Q)|$ for every $P$ and $Q$ on $\ell$ is called a *coordinate function* for the line $\ell$ and the number $f(P)$ is called the *coordinate of $P$*.

**Theorem 5.4.14 (Ruler Placement):** For every pair of distinct points $P$ and $Q$, there is a coordinate function $f : \overrightarrow{PQ} \to \mathbb{R}$ such that $f(P) = 0$ and $f(Q) > 0$. 
Def: A set of points $S$ is a *convex set* if for every pair of points $A, B \in S$, $AB \subset S$.

Axiom 5.5.2 (Plane Separation Postulate): For every line $\ell$, the points that do not lie on $\ell$ form two disjoint, nonempty sets $H_1$ and $H_2$, called *half-planes bounded by* $\ell$, such that

1. Each of $H_1$ and $H_2$ is convex.
2. If $P \in H_1$ and $Q \in H_2$, then $PQ$ intersects $\ell$.

Notation: Given $\ell$ and external point $A$, $H_A$ is the half-plane bounded by $\ell$ containing $A$.

Def: Let $\ell$ be a line with external points $A$ and $B$. $A$ and $B$ are *on the same side of* $\ell$ if they are both in $H_1$ or both in $H_2$. They are on *opposite sides of* $\ell$ if one is in $H_1$ and the other is in $H_2$. 
Def: Two rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are opposite rays if they are not equal and $B \neq A \neq C$.

Def: An angle is the union of two nonopposite rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$. The point $A$ is the vertex, and the rays are the sides.

Def: Interior of an angle, collinear, triangle.
Pasch’s Theorem: Let $\triangle ABC$ be a triangle and let $\ell$ be a line such that none of $A$, $B$, and $C$ lies on $\ell$. If $\ell$ intersects $\overline{AB}$, the $\ell$ also intersects either $\overline{AC}$ or $\overline{BC}$. 
Axiom 5.6.2 (The Protractor Postulate): For every angle \( \angle BAC \) there is a real number \( \mu(\angle BAC) \), called the measure of \( \angle BAC \), such that the following conditions are satisfied.

1. \( 0^\circ \leq \mu(\angle BAC) < 180^\circ \) for every angle \( \angle BAC \).
2. \( \mu(\angle BAC) = 0^\circ \) if and only if \( \overrightarrow{AB} = \overrightarrow{AC} \).
3. (Angle Construction Postulate) For each real number \( r, 0 < r < 180 \), and for each half-plane \( H \) bounded by \( \overrightarrow{AB} \) there exists a unique ray \( \overrightarrow{AE} \) such that \( E \) is in \( H \) and \( \mu(\angle BAE) = r^\circ \).
4. (Angle Addition Postulate) If the ray \( \overrightarrow{AD} \) is between rays \( \overrightarrow{AB} \) and \( \overrightarrow{AC} \), then

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\mu(\angle BAD) + \mu(\angle DAC) + \mu(\angle BAC)
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