$1 - 1/2 + 1/3 - 1/4 + \cdots + 1/2003 - 1/2004 = 1/1003 + 1/1004 + \cdots + 1/2004$: A Fortnight of Induction and Verse

by Aaron Cinzori

A fortnightly problem involving addition,
subtraction, some fractions, and my intuition.
I'd worked on such problems at times in the past
(they appear on the Putnam where you have to work fast)
but I'd never solved one. I blame the time pressure,
but this one I'd be able to work at my leisure.
Additionally there was the problem of verse,
but that should be easy. The math would be worse.
That, at least, was my thought at the start.

The first thing I saw was the number of fractions:
The left side twice right. So my early reactions
involved pairing pairs of the left hand side stuff
with terms from the right that were equal—not tough.
For example 1 over 1002
minus 1 over 2004, as I knew,
is 1 over 2004. Oh, but wait,
that 1 over 1002's minus. Great.
I was right. It's the math's the hard part.

The other false starts I endured need no mention,
but an old rule of thumb finally got my attention:
Gen'ralization is frequently key in the case
of a concrete example that's hard on its face.
You broaden the scope of your investigation.
The problem at hand's but an instantiation
of the general problem with patterns that yield
to the tools mathematicians tend to use in their field.
Solution? Induction. Begin:

Define A sub n as a harmonic sum
that ends with $2n$ and begins with a one,
its signs alternate and the odd terms are pluses.
With that now in place, the next thing we discuss is
defining S_n. It's a harmonic sum from
1 up to $2n$ minus one that's begun 1
and continues until we get n. Now it's true
that the problem (when n is 1002)
is solved if A_n is S_n.

That A_1 is S_1 is clear by inspection.
Now proceed by induction to forge the connection.
The inductive hypothesis says that A_n
is S_n. So step up and do it again.
Since A_{n+1} is A_n with a pair
of fractions subtracted and added back there,
we use the hypothesis, swap S for A,
then first and last terms sum together and, say,
well hurray, we have S_{n+1}.

So that means our proof is now done.
(And the verse was the harder, though fun.)