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THE ANSWER IS 1 - 1/e. WHAT IS THE QUESTION?
(Two matching problems that have the same limiting answer.)

by Elliot A. Tanis
Hope College

An urn contains $n$ balls numbered from 1 through $n$. The balls are selected randomly from the urn, one at a time, until $n$ balls have been selected. A match occurs if ball numbered $k$ is the $k^{th}$ ball that is selected. We are interested in finding the probability that there is at least one match.

Does selecting the balls with replacement or selecting without replacement affect the probability of at least one match? Does the number of balls in the urn affect the probability of at least one match? You should be able to answer these questions after you have read this paper.

**Problem 1.** An urn contains $n$ balls numbered 1 through $n$. From the urn $n$ balls are selected one at a time with replacement. A match occurs if ball numbered $k$ is the $k^{th}$ ball that is selected. Find the probability of at least one match, say $q_n$.

**Problem 1'.** Roll an $n$-sided die $n$ times. A match occurs if side $k$ is the outcome on the $k^{th}$ roll, $k = 1, 2, \ldots, n$. Find the probability of at least one match during the $n$ rolls of the die, say $q_n$.

**Solution.** We find the probability of no matches and subtract this answer from 1. The probability of no matches is easy to calculate because the trials are independent. We have

$$q_n = P(\text{at least one match}) = 1 - P(\text{no matches})$$

$$= 1 - \left(\frac{n - 1}{n}\right)\left(\frac{n - 1}{n}\right) \cdots \left(\frac{n - 1}{n}\right)$$

$$= 1 - \left(\frac{n - 1}{n}\right)^n = 1 - (1 - \frac{1}{n})^n$$
Note that as \( n \) increases without bound,

\[
\lim_{n \to \infty} q_n = \lim_{n \to \infty} \left[ 1 - \left(1 - \frac{1}{n}\right)^n \right] = 1 - \frac{1}{e}
\]

and thus we can write a question for which the answer is \( 1 - 1/e \).

**Problem 2.** An urn contains \( n \) balls numbered from 1 through \( n \). From the urn \( n \) balls are selected one at a time without replacement. A match occurs if ball numbered \( k \) is the \( k \)th ball selected. (Note that this generates a random permutation of the first \( n \) positive integers.) Find the probability of at least one match, say \( p_n \).

**Problem 2'.** Let \( A \) and \( B \) denote two identical decks of cards, each deck containing \( n \) cards numbered from 1 through \( n \). Shuffle each deck. A match occurs if card numbered \( k \), \( 1 \leq k \leq n \), occupies the same position in each deck. Find the probability of at least one match, say \( p_n \).

**Solution (when \( n=4 \)).** Let the event \( A_i \) denote a match on the \( i \)th draw. Then

\[
P(A_i) = \frac{3!}{4!}
\]

\[
P(A_i \cap A_j) = \frac{2!}{4!}
\]

\[
P(A_i \cap A_j \cap A_k) = \frac{1!}{4!}
\]

\[
P(A_i \cap A_j \cap A_k \cap A_{\ell}) = \frac{1}{4!}
\]

The probability of at least one match is

\[
P(A_1 \cup A_2 \cup A_3 \cup A_4) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - P(A_1 \cap A_2 \cap A_3 \cap A_4)
\]

\[
= 4 \left( \frac{3!}{4!} \right) - \left( \binom{4}{2} \right) \left( \frac{2!}{4!} \right) + \left( \binom{4}{3} \right) \left( \frac{1}{4!} \right) - \frac{1}{4!}
\]

\[
= \frac{4!}{4!} - \frac{4!}{2! \cdot 3!} \cdot \frac{2!}{4!} + \frac{4!}{3! \cdot 4!} \cdot \frac{1}{4!} - \frac{1}{4!}
\]

\[
= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!}
\]

The solution for any integer \( n \) is

\[
p_n = P(\text{at least one match})
\]

\[
= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \ldots + \frac{(-1)^{n+1}}{n!}
\]
As $n$ increases without bound,

$$\lim_{n \to \infty} p_n = 1 - \left[ 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \cdots \right] = 1 - \frac{1}{e}$$

so we can write a second question for which the answer is $1 - 1/e$.

It is interesting to compare the values of $p_n$ and $q_n$ for different values of $n$. We can construct the following table.

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<th>$p_n$</th>
<th>$q_n$</th>
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</tr>
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Recall that

$$1 - \frac{1}{e} \approx 0.63212$$

We see that the value of $n$ has little affect on $p_n$ when $n > 4$

and on $q_n$ when $n > 15$. Also for large $n$, $p_n$ and $q_n$ are approximately equal.

An interesting exercise would be to illustrate these probabilities empirically. Either use dice, cards, or balls in an urn - or, write a program to simulate these experiments on a computer.
THE ANSWER IS $1 - \frac{1}{e}$. WHAT IS THE QUESTION?

(Two matching problems that have the same limiting answer.)

Elliot A. Tanis
Hope College
October 19, 1985

Problem 1. Roll an $n$-sided die $n$ times. A match occurs if side $k$ is the outcome on the $k$'th roll, $k = 1, 2, \ldots, n$. Find the probability of at least one match during the $n$ rolls of the die, say $q_n$.

Problem 1'. An urn contains $n$ balls numbered 1 through $n$. From the urn $n$ balls are selected one at a time with replacement. A match occurs if ball numbered $k$ is the $k$'th ball that is selected. Find the probability of at least one match, say $q_n$.

Solution: $q_n = P($at least one match$)$

$= 1 - P($no matches$)$

$= 1 - \left(\frac{n-1}{n}\right) \cdot \frac{n-1}{n} \cdots \frac{n-1}{n}$

$= 1 - \left(\frac{n-1}{n}\right)^n = 1 - \left(1 - \frac{1}{n}\right)^n$

$\lim_{n \to \infty} q_n = \lim_{n \to \infty} \left[1 - \left(1 - \frac{1}{n}\right)^n\right] = 1 - \frac{1}{e}$

10 CLS:KEY OFF: SCREEN 1
20 PRINT "An urn contains $n$ balls numbered "
30 PRINT "1 through $n$. These balls are selected"
40 PRINT "from the urn, one at a time. A match"
50 PRINT "occurs if ball numbered $k$ is the $k$'th"
60 PRINT "ball that is selected. Find empirically"
70 PRINT "P(at least one match)"
80 PRINT "if the sampling is done with and/or"
90 PRINT "without replacement."
100 PRINT PRINT PRINT PRINT "Input 0 for without replacement." PRINT PRINT "Input 1 for with replacement. " PRINT INPUT REP
110 IF REP = 0 THEN 500
200 CLS:KEY OFF: SCREEN 1
210 RANDOMIZE TIMER
220 M = 100: REM The number of repetitions
225 PRINT "Sampling WITH replacement":PRINT
230 PRINT "This program simulates N rolls of an N-sided die. A match occurs if side k is observed on the k’th roll. A trial (N rolls) is successful if at least one match is observed. The experiment is"
240 PRINT "repeated M =";M,"times. The program":PRINT "counts the number of successful trials."
250 PRINT :PRINT "INPUT N = ":INPUT N
260 FOR K = 1 TO M
270 PRINT K " - ";
280 FOR L = 1 TO N
290 D = INT(N*RND) + 1
300 FLAG(D) = 1
310 IF L =D THEN S = 1
320 PRINT D;
330 NEXT L
340 PRINT " - ";
350 IF S = 1 THEN PRINT "yes"
360 IF S = 0 THEN PRINT "no"
370 NS = NS + S
380 S = 0
390 NEXT K
400 PRINT :PRINT "The number of trials that had at least one match is ";NS
410 PRINT:PRINT "The proportion of trials on which at least one match was observed is ";NS/M
420 PRINT
430 ON = 1 - (1 - 1/N)^N
440 PRINT "The probability of at least one match"
450 PRINT "is ";ON
460 END
Sampling WITH replacement

This program simulates N rolls of an N-sided die. A match occurs if side k is observed on the k’th roll. A trial (N rolls) is successful if at least one match is observed. The experiment is repeated M = 100 times. The program counts the number of successful trials.

**INPUT N = 8**

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<th>8</th>
<th>6</th>
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</table>

| 88 | 6 | 1 | 8 | 4 | 3 | 3 | 1 | 1 |
| 89 | 6 | 5 | 7 | 2 | 5 | 2 | 3 | 7 |
| 90 | 7 | 6 | 1 | 2 | 2 | 6 | 1 | 1 |
| 91 | 5 | 7 | 8 | 7 | 6 | 5 | 7 | 3 |
| 92 | 4 | 5 | 2 | 6 | 6 | 3 | 2 | 8 |
| 93 | 1 | 4 | 2 | 2 | 4 | 4 | 4 | 5 |
| 94 | 6 | 7 | 4 | 8 | 6 | 3 | 3 | 5 |
| 95 | 1 | 3 | 4 | 4 | 6 | 5 | 1 | 5 |
| 96 | 5 | 4 | 6 | 1 | 2 | 5 | 4 | 1 |
| 97 | 6 | 5 | 6 | 4 | 5 | 8 | 7 | 4 |
| 98 | 8 | 3 | 1 | 1 | 3 | 2 | 6 | 3 |
| 99 | 7 | 1 | 1 | 3 | 1 | 6 | 5 | 2 |
| 100 | 7 | 5 | 4 | 7 | 1 | 1 | 8 | 4 |

The number of trials that had at least one match is 67

The proportion of trials on which at least one match was observed is .67

The probability of at least one match is .6563911

Ok
Problem 2. Let A and B denote two identical decks of cards, each deck containing \( n \) cards numbered from 1 through \( n \). Shuffle each deck. A match occurs if card numbered \( k \), \( 1 \leq k \leq n \), occupies the same position in each deck. Find the probability of at least one match, say \( p_n \).

Problem 2'. An urn contains \( n \) balls numbered from 1 through \( n \). From the urn \( n \) balls are selected one at a time without replacement. A match occurs if ball numbered \( k \) is the \( k \)'th ball selected. (Note that this generates a random permutation of the first \( n \) positive integers.) Find the probability of at least one match, say \( p_n \).

Solution (when \( n = 4 \)):

Let the event \( A_i \) denote a match on the \( i \)'th draw. Then

\[
P(A_1) = \frac{31}{41} \]

\[
P(A_1 \cap A_j) = \frac{21}{41} \]

\[
P(A_1 \cap A_j \cap A_k) = \frac{11}{41} \]

\[
P(A_1 \cap A_2 \cap A_3 \cap A_4) = \frac{1}{41} \]

\[
P(A_1 \cup A_2 \cup A_3 \cup A_4) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_1 \cap A_j \cap A_k) - P(A_1 \cap A_2 \cap A_3 \cap A_4) \]

\[
= 4\left(\frac{31}{41}\right) - \binom{4}{2}\left(\frac{21}{41}\right) + \binom{4}{3}\left(\frac{11}{41}\right) - \frac{1}{41} \]

\[
= \frac{41}{41} - \frac{41}{2 \cdot 21} \frac{21}{41} + \frac{41}{2 \cdot 3 \cdot 11} \frac{1}{41} - \frac{1}{41} \]

\[
= 1 - \frac{1}{21} + \frac{1}{31} - \frac{1}{41} \]

The general solution is

\[
p_n = P(\text{at least one match}) = 1 - \frac{1}{21} + \frac{1}{31} - \frac{1}{41} + \ldots + \frac{(-1)^{n+1}}{n!} \]

\[
\lim_{n \to \infty} p_n = 1 - [1 - 1 + \frac{1}{21} - \frac{1}{31} + \frac{1}{41} - \frac{1}{51} + \ldots] = 1 - \frac{1}{e} \]
CLS: KEY OFF: SCREEN 1
510 RANDOMIZE TIMER
520 DIM FLAG(20), N(20)
530 M = 100: REM NUMBER OF REPETITIONS
535 PRINT "Sampling WITHOUT replacement":PRINT
540 PRINT "This program generates M = " ; M; " random":PRINT "permutations of the first N positive integers. It counts the number of"
550 PRINT "permutations for which at least one integer is in its natural position."
560 PRINT :PRINT "INPUT N = "; :INPUT N
570 FOR K = 1 TO M
580 PRINT K "- ";
590 FOR J = 1 TO N
600 FLAG(J) = 0
610 NEXT J
620 FOR L = 1 TO N
630 D = INT(N*RND) + 1
640 IF FLAG(D) = 1 THEN GOTO 630
650 N(L) = D
660 FLAG(D) = 1
670 IF L = D THEN S = 1
680 PRINT D;
690 NEXT L
700 PRINT "- ";
710 IF S = 1 THEN PRINT "yes"
720 IF S = 0 THEN PRINT "no"
730 NS = NS + S
740 S = 0
750 NEXT K
760 PRINT :PRINT "The number of trials that had at least one match is "; NS
770 PRINT :PRINT "The proportion of trials on which at least one match was observed is "; NS/M
780 PRINT
790 DEN = 1: PN = 1
800 FOR K = 2 TO N
810 DEN = DEN * K
820 PN = PN + ((-1)^(K+1))/DEN
830 NEXT K
840 PRINT "The probability of at least one match"
850 PRINT "is "; PN
860 END
Sampling WITHOUT replacement

This program generates $M = 100$ random permutations of the first $N$ positive integers. It counts the number of permutations for which at least one integer is in its natural position.

**INPUT N = ? 8**

1 - 3 7 2 1 8 5 4 6 - no  
2 - 7 4 8 2 5 6 1 3 - yes  
3 - 6 8 4 2 1 3 5 7 - no  
4 - 1 2 5 7 3 6 4 8 - yes  
5 - 4 6 5 3 1 2 8 7 - no  
6 - 4 8 7 6 2 1 3 5 - no  
7 - 8 4 7 2 6 1 5 3 - no  
8 - 1 8 5 6 7 2

88 - 5 2 4 7 6 3 1 8 - yes  
89 - 4 1 2 8 5 7 3 6 - yes  
90 - 7 8 5 2 3 1 6 4 - no  
91 - 1 6 2 3 7 4 5 8 - yes  
92 - 5 4 2 7 1 8 6 3 - no  
93 - 6 5 4 3 8 1 7 2 - yes  
94 - 3 7 2 4 6 5 1 8 - yes  
95 - 6 2 3 5 7 1 4 8 - yes  
96 - 1 7 2 8 6 3 4 5 - yes  
97 - 7 5 8 1 2 4 6 3 - no  
98 - 4 5 7 8 6 1 3 2 - no  
99 - 5 2 3 7 6 8 4 1 - yes  
100 - 4 5 6 1 7 3 2 8 - yes

The number of trials that had at least one match is **60**

The proportion of trials on which at least one match was observed is **.6**

The probability of at least one match is **.6321181**

Ok
CLS:KEY OFF: SCREEN 2
20 LPRINT " N       Pn         On"
30 DIM PN(15),QN(15),DEN(15)
40 N = 1
50 QN(1) = 1
60 PN(1) = 1
70 DEN(1) = 1
80 LPRINT N,PN(1),QN(1)
90 FOR N = 2 TO 15
100 DEN(N) = N*DEN(N-1)
110 PN(N) = PN(N-1) + ((-1)^N+1)/DEN(N)
120 QN(N) = 1 - (1 - 1/N)^N
130 LPRINT N,PN(N),QN(N)
140 NEXT N
150 END

<table>
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<tr>
<th>N</th>
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<th>Qn</th>
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</table>

\[ 1 - \frac{1}{e} = 0.632120558 \]
THE ANSWER IS 1 - 1/e. WHAT IS THE QUESTION?
(Two matching problems that have the same limiting answer.)

Elliot A. Tanis
Hope College
Holland, Michigan 49423

An urn contains \( n \) balls numbered from 1 through \( n \). The balls are selected randomly from the urn, one at a time, until \( n \) balls have been selected. A match occurs if ball numbered \( k \) is the \( k^{th} \) ball that is selected. We are interested in finding the probability that there is at least one match.

Does selecting the balls with replacement or selecting without replacement affect the probability of at least one match? Does the number of balls in the urn affect the probability of at least one match? You should be able to answer these questions after you have read this paper.

Problem 1. An urn contains \( n \) balls numbered 1 through \( n \). From the urn \( n \) balls are selected one at a time with replacement. A match occurs if ball numbered \( k \) is the \( k^{th} \) ball that is selected. Find the probability of at least one match, say \( q_n \).

Problem 1'. Roll an \( n \)-sided die \( n \) times. A match occurs if side \( k \) is the outcome on the \( k^{th} \) roll, \( k = 1, 2, \ldots, n \). Find the probability of at least one match during the \( n \) rolls of the die, say \( q_n \).

Solution: We find the probability of no matches and subtract this answer from 1. The probability of no matches is easy to calculate because the trials are independent. We have

\[
q_n = P(\text{at least one match})
\]
\[
= 1 - P(\text{no matches})
\]
\[
= 1 - \left( \frac{n-1}{n} \right) \left( \frac{n-1}{n} \right) \cdots \left( \frac{n-1}{n} \right)
\]
\[
= 1 - \left( \frac{n-1}{n} \right)^n = 1 - \left( 1 - \frac{1}{n} \right)^n
\]

Note that as \( n \) increases without bound,

\[
\lim_{n \to \infty} q_n = \lim_{n \to \infty} \left[ 1 - \left( 1 - \frac{1}{n} \right)^n \right] = 1 - \frac{1}{e}
\]

and thus we can write a question for which the answer is \( 1 - 1/e \).

Problem 2. An urn contains \( n \) balls numbered from 1 through \( n \). From the urn \( n \) balls are selected one at a time without replacement. A match occurs if ball numbered \( k \) is the \( k^{th} \) ball selected. (Note that this generates a random permutation of the first \( n \) positive integers.) Find the probability of at least one match, say \( p_n \).
Problem 2'. Let A and B denote two identical decks of cards, each deck containing n cards numbered from 1 through n. Shuffle each deck. A match occurs if card numbered k, 1 ≤ k ≤ n, occupies the same position in each deck. Find the probability of at least one match, say \( p_n \).

Solution (when n=4): Let the event \( A_i \) denote a match on the \( i^{th} \) draw. Then

\[
P(A_i) = \frac{3!}{4!} \\
P(A_i \cap A_j) = \frac{2!}{4!} \\
P(A_i \cap A_j \cap A_k) = \frac{1!}{4!} \\
P(A_1 \cap A_2 \cap A_3 \cap A_4) = \frac{1}{4!}
\]

The probability of at least one match is

\[
P(A_1 \cup A_2 \cup A_3 \cup A_4) = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - P(A_1 \cap A_2 \cap A_3 \cap A_4)
\]

\[
= 4 \left( \frac{3!}{4!} \right) - \left( \frac{4}{2} \right) \left( \frac{2!}{4!} \right) + \left( \frac{4}{3} \right) \left( \frac{1}{4!} \right) - \frac{1}{4!}
\]

\[
= \frac{4!}{4!} - \frac{4!}{2!2!4!} + \frac{4!}{3!1!4!} - \frac{1}{4!}
\]

\[
= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!}
\]

The solution for any integer n is

\[
p_n = P(\text{at least one match})
\]

\[
= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \ldots + \frac{(-1)^{n+1}}{n!}
\]

As n increases without bound,

\[
\lim_{n \to \infty} p_n = 1 - \left[ 1 - \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \ldots \right] = 1 - \frac{1}{e}
\]

so we can write a second question for which the answer is 1 - 1/e.

It is interesting to compare the values of \( p_n \) and \( q_n \) for different values of n. We can construct the following table.
<table>
<thead>
<tr>
<th>n</th>
<th>$p_n$</th>
<th>$q_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.00000</td>
<td>1.00000</td>
</tr>
<tr>
<td>2</td>
<td>0.50000</td>
<td>0.75000</td>
</tr>
<tr>
<td>3</td>
<td>0.66667</td>
<td>0.70370</td>
</tr>
<tr>
<td>4</td>
<td>0.62500</td>
<td>0.68359</td>
</tr>
<tr>
<td>5</td>
<td>0.63333</td>
<td>0.67232</td>
</tr>
<tr>
<td>6</td>
<td>0.63194</td>
<td>0.66510</td>
</tr>
<tr>
<td>7</td>
<td>0.63214</td>
<td>0.66008</td>
</tr>
<tr>
<td>8</td>
<td>0.63212</td>
<td>0.65639</td>
</tr>
<tr>
<td>9</td>
<td>0.63212</td>
<td>0.65356</td>
</tr>
<tr>
<td>10</td>
<td>0.63212</td>
<td>0.65132</td>
</tr>
<tr>
<td>11</td>
<td>0.63212</td>
<td>0.64951</td>
</tr>
<tr>
<td>12</td>
<td>0.63212</td>
<td>0.64800</td>
</tr>
<tr>
<td>13</td>
<td>0.63212</td>
<td>0.64674</td>
</tr>
<tr>
<td>14</td>
<td>0.63212</td>
<td>0.64566</td>
</tr>
<tr>
<td>15</td>
<td>0.63212</td>
<td>0.64474</td>
</tr>
</tbody>
</table>

Recall that

$$1 - \frac{1}{e} \approx 0.63212$$

We see that the value of $n$ has little affect on $p_n$ when $n > 4$ and on $q_n$ when $n > 15$. Also for large $n$, $p_n$ and $q_n$ are approximately equal.

An interesting exercise would be to illustrate these probabilities empirically. Either use dice, cards, or balls in an urn — or, write a program to simulate these experiments on a computer.