THE COMPUTER AS AN INTERFACE BETWEEN MATHEMATICS AND ART

Elliot A. Tanis

Hope College
Holland, Michigan 49423
United States of America

The computer with graphics capabilities provides a new tool for the artistically inclined student. In order to use the computer to create artistic designs, the student gains an understanding of certain mathematical concepts, an increased ability in using the computer, and a greater appreciation of art.

The author has worked with several college level mathematics students on projects that used the computer for artistic purposes. This paper discusses three of these projects that were related to repeating patterns.

The Hope College Art Gallery was used to exhibit 22 of our computer drawings. The public gained a new appreciation for the computer when viewing this exhibit.

1. INTRODUCTION

A computer with graphics capabilities provides an excellent vehicle for artistic purposes. In addition, using the computer for artistic purposes provides an opportunity for a student to learn some mathematical concepts.

The author has worked with five students during the past four years on projects in which the computer has been used for artistic purposes. Each student [1-5] gave an oral report of his project at the Annual Meeting of the Michigan Section of the Mathematical Association of America. The author also presented an hour address at one of those meetings [6]. In addition, parts of two projects have been published [7,8].

In this paper we discuss the work of three of these students. In particular we discuss repeating patterns motivated by the graphic works of M. C. Escher.

The Dutch graphics artist, Maurits C. Escher (1898-1972) [9-13] was intrigued by repeating patterns and the idea of filling the plane with figures of animals that leave no gaps and do not overlap. He was originally inspired by the Moorish mosaics in the Alhambra in Granada, Spain, and in the mosque, LaMezquita, in Córdoba.

Many people, including mathematicians, have been intrigued by the works of M. C. Escher. With little formal training in mathematics, Escher displayed a real understanding of the mathematical concepts required for his drawings to "work."

2. REPEATING PATTERNS

An excellent mathematical discussion of repeating patterns or the plane symmetry groups is contained in an article by Doris Schattschneider [15]. For our purposes think of a repeating pattern or tessellation as a tiling of the plane. A unit cell (tile) of the pattern is the smallest region of the plane which has the property that the set of all of its images covers the plane. These tiles, when laid in parallel rows, cover the plane.

Associated with a tessellation is a set of lattice points. These are often the vertices of the unit cell. Lattice points will be illustrated in the examples.

There are four types of transformations in the plane that are necessary for studying and classifying tile patterns. They are translations, rotations, reflections, and glide reflections. These will also be illustrated in the examples.

Crystallographers have developed a system for classifying crystals according to their symmetries. Repeating patterns in the plane can be classified using the two-dimensional counterpart of their system. Examples are given in [15].

It can be shown that there are 17 distinct plane symmetry groups. These are summarized in [15]. We shall give illustrations of seven of these groups.

3. ROLE OF THE COMPUTER

We studied several of M. C. Escher's tesselations. Six books in the references [9-14] contain examples of Escher's tessellations. We also studied some unpublished tesselations at the Escher Foundation in the Gemeentemuseum in the Haag. For each of Escher's drawings we determined the symmetry group to which it belongs and for several of them, we simulated the drawing on the computer. We also studied some Islamic art designs and created some original drawings.
In order to have the computer make the appropriate drawing, several programs had to be written. All of our computer programs are written in BASIC. They were written on a Tektronix 4051 graphics computer and the drawings were made on the Tektronix 4662 interactive digital plotter. We have recently purchased an Apple II computer and hope to adapt our programs so that they can be run on the Apple II.

The computer programs that we have written will be illustrated with some examples in the following section. We drew them in color on the plotter. Because this paper is printed in black and white, the reader will have to supply the color.

4. EXAMPLES

The subheadings in this section are the crystallographic notation for each design. The full notation is given first. If there is also a short form, it is given in parentheses.

4.1 P1

The simplest type of tessellation is that based on translation and having no symmetries. The birds in Figure 1, patterned after an Escher drawing, illustrate this.

The computer program required to draw this pattern is quite simple. All of the information in a tile must be input. A tile has been outlined in Figure 1. In order to minimize the number of times that the pen would be raised and lowered, we input this information as two arrays of numbers that gave the coordinates of 14 points on the left side of the bird and 10 points on the top of the bird. These points are connected with straight line segments to create the motif.
A lattice for this design is the tops of the heads of the birds. Note that the vertices of the tile give four lattice points. The program moves from lattice point to lattice point, left to right, in each row. At each lattice point it draws the motif. After it completes a row it moves up one row and repeats the process.

An inner array has been defined for each bird in Figure 2. Coloring in every other bird makes it easier to see them.

4.2 P211 (P2)

The doves in Figure 3, patterned after an Escher drawing, illustrate 2-fold rotation. A unit cell or tile is again outlined, this time as a parallelogram. Note that if the pattern is rotated 180° about one of the vertices of this parallelogram, the entire pattern is transformed onto itself. Similarly rotating the pattern 180° about the center of the parallelogram or about the midpoint of any side of the parallelogram also transforms the pattern onto itself. These nine rotation points are called points of 2-fold rotation.

For this type of pattern, half of the information in a tile must be provided. We provided the information given as the motif by defining 28 points that were then connected with straight line segments.

The vertices of the parallelogram give four of the lattice points for this pattern. The program again moves from lattice point to lattice point in a row and then moves up to the next row. At each lattice point the motif is drawn as given. The motif is then rotated 180° about the lattice point and drawn again.

In Figure 4 an inner array has been added for
each dove. Again coloring in every other dove will make it easier to distinguish one dove from an adjacent dove.

4.3 P1M1 (PM)

There are no rotations present in the pattern of fish in Figures 5 and 6. However those fish illustrate reflection or mirror lines. A unit cell or tile has been outlined in Figure 5. Note that the pattern can be reflected onto itself by reflecting about either of the horizontal sides of the rectangle or about a line that intersects the midpoints of the vertical sides of the rectangle.

The motif that was input to generate this pattern in one half of a fish. A set of lattice points includes the vertices of the unit cell that has been outlined.

For this pattern the motif is drawn at each lattice point. The motif is then reflected about a horizontal line through the lattice points and is drawn again. The program moves from lattice point to lattice point, left to right, in a row. When a row has been completed, it moves up to the next row, beginning at the left.

Figure 6 shows the fish with some interior arrays. Note that the fish going from right to left are different from the fish that go from left to right. The reader is encouraged to color the fish going from right to left in order to distinguish them from the other fish.

4.4 P111 (PG)

Figure 7 illustrates the fourth transformation in the plane, namely, a glide reflection. This design is patterned after Fischer’s drawing on page 66 of [13]. A unit cell has again been out-

![P1M1 Motif](image1)

![P1M1 Motif](image2)

![Figure 5](image3)

![Figure 6](image4)
lined, this time as a rectangle. The amount of information that must be input is that contained within the left half of the rectangle. This has been input as the motif. Part of the motif is drawn outside of the cell in order that the motif can be input without requiring the pen to be raised.

The computer program moves the pen to a lattice point, for example, the lower left hand corner of the outlined unit cell. The motif is drawn. To handle the glide reflection, the motif is translated horizontally to the right, one half the distance of the unit cell. This is the glide. The motif is then reflected about a horizontal line through the center of the unit cell and this reflected motif is drawn.

To draw the entire design the above process is repeated moving left to right from lattice point to lattice point. When a row has been completed, the next row up is drawn.

Some inner arrays have been added to the frogs in Figure 8. Because the frogs are not symmetrical, all of the inner arrays must be input.

The reader is again encouraged to color every other frog so that they are easier to distinguish.

4.5 P31M

This example, based on Escher's drawing on page 81 of [13], illustrates 3-fold rotations, that is, 3 rotations of 120º about certain points. The unit cell in this example is a parallelogram composed of two equilateral triangles.

Looking at the unit cell we see that the entire design can be reflected about an edge of this parallelogram or about a line through the short-
er diagonal.

There are six points of 3-fold rotation, namely, the four vertices of the unit cell and at the two trisection points on the longer diagonal of the unit cell. The entire design can be rotated 120° or 240° about any of these points onto itself.

The computer program moves the pen to a lattice point, for example, the lower left hand corner of the outlined unit cell. The motif, that has been defined using five points connected with straight line segments, is then drawn. This motif is rotated 120° and drawn again. It is then rotated an additional 120° and drawn a third time. The motif is rotated an additional 120° back to its original position and is then reflected about one side of the unit cell. The reflected motif is drawn, rotated 120° and drawn again, and rotated an additional 120° and drawn for a third time.

The above procedure is repeated at each of the lattice points in turn in a systematic order.

Some inner arrays have been added in Figure 10. Because of the symmetry of these flukes, only half of the inner arrays have to be defined.

In order to separate these flukes by coloring in some of them, note that three colors are required. The reader is again encouraged to do this.

4.6 P31m

Figure 11 gives an example of an Islamic art pattern located in the Kashan region of Iran, from the 13th-14th century. The original tiling includes Persian poetry and additional designs. M. C. Escher did not make a tessellation be-
longing to this particular plane symmetry group.

A square unit cell has been outlined. This design illustrates 4-fold rotations. The motif is rotated 90° four times successively about a point of 4-fold rotation. Points of 4-fold rotation occur at the vertices and the center of the unit cell. In addition there are points of 2-fold rotation at the midpoints of the sides of the unit cell. In this pattern there are several reflection lines, namely, the four sides of the square, the two diagonals, and the horizontal and vertical bisectors of the sides of the square. Note that there are also lines of glide reflection.

The computer program moves the pen to a lattice point. The motif is drawn and then successively rotated 90° about the lattice point and drawn three times for a total of four times. The motif is then reflected about the 45° line through the lattice point and this reflected motif is drawn. The reflected motif is then successively rotated 90° about the lattice point and drawn three additional times.

The above procedure is repeated at each lattice point in a systematic manner.

4.7 P6MM

Figure 12 gives an example of an Islamic art pattern based on a molded tile panel in Nishapur from the 13th-14th century. The tiling includes designs on the stars and hexagons.

The unit cell is a parallelogram composed of two equilateral triangles. There are points of 6-fold rotation at the vertices of the unit cell. At these points the entire design can be rotated six times successively 60° onto itself. There are points of 3-fold rotation at the trisection
points on the long diagonal. There are points of 2-fold rotation at the midpoints of the four sides of the unit cell and the center of the unit cell. There are many reflection lines in this pattern and they are indicated on Figure 12 in the upper right hand corner.

Only two points are required to define the motif for this pattern. The computer program moves the pen to a lattice point, for example the lower left hand corner of the parallelogram, and draws this motif. It then successively rotates the motif 60° and draws it. After the motif has been drawn six times, it is reflected about the 30° line through the lattice point. The reflected motif is then drawn and successively rotated 60° and drawn for a total of six times. This procedure is repeated at each lattice point in a systematic order.

5. SUMMARY OF REPEATING PATTERNS

We have written computer programs in BASIC for drawing examples of each of the 17 possible plane symmetry groups. It is not possible in a paper of this length to give examples of each of them. However, the author is willing to send examples of each of the 17 groups to the interested reader.

6. CONCLUSIONS

There are many ways in which the computer can be used for artistic purposes. This paper discusses one area, that of repeating patterns. An understanding of the mathematical properties of repeating patterns is very useful in analyzing the patterns and developing original drawings.

The computer forms an effective interface between mathematics and art. Projects like those given in this paper are especially appropriate for students in a liberal arts college where students take courses in both mathematics and art.

Microcomputers like the Apple II could provide an effective and relatively inexpensive tool for the mathematically inclined art student or the artistically inclined mathematics student to develop some interesting designs and artistic works.

7. ACKNOWLEDGEMENTS

The author wishes to thank Dr. Willard C. Wichers, Netherlands Consul for Press and Cultural Affairs in Holland, Michigan, for obtaining permission for the author to study the original works of M.C. Escher at the Escher Foundation in the Hague Gemeentemuseum and Dr. J. L. Locher, Director of the Escher Foundation, for his warm reception and generous help. In addition the author wishes to thank his colleagues and the administration of Hope College for encouragement and support.

REFERENCES